The Interior and Closure of Strongly Stable Matrices

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ABSTRACT

Necessary and sufficient conditions for a matrix to be in the interior and closure of the set of strongly stable matrices are presented.

A real square matrix A is stable if the real part of each eigenvalue of A is positive, and strongly stable if $A + D$ is stable for any nonnegative diagonal matrix D. The concept of strong stability arises when diffusion models of biological systems are linearized at a constant equilibrium [3]. Recently, some properties of this class of matrices have been studied [2, 5, 7]. Here we will discuss the characterizations of its interior and closure.

Let M_n denote the set of real $n \times n$ matrices, SM_n denote the set of $n \times n$ stable matrices, and SSM_n denote the set of $n \times n$ strongly stable matrices. A (strongly) stable matrix is said to be totally (strongly) stable if all its principal submatrices are (strongly) stable. $A \in \mathrm{M}_n$ is said to be semistable if the real part of each eigenvalue of A is nonnegative. $A \in \mathrm{M}_n$ is said to be strongly semistable if $A + D$ is semistable for any nonnegative diagonal matrix D. Let SM_n^+ (SSM_n^+) and SM_n^0 (SSM_n^0) denote the sets of $n \times n$ totally (strongly) stable matrices and $n \times n$ (strongly) semistable matrices, respectively.

We note that the set SSM_n is neither open nor closed (with respect to the usual topology), because

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 2 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

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is in SSM_n, but

$$A_\epsilon=\begin{pmatrix} \epsilon & 1 & 0 & \dots & 0 \\ -1 & 2 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \to A$$

is not in SSM_n for $\epsilon < 0$, and

$$B=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

is not in SSM_n, but

$$B_\delta=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & \delta & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \to B$$

is in SSM_n for $\delta > 0$. This can easily be verified by using the fact that a 2×2 matrix is strongly stable if and only if it belongs to P_0^+ [2].

For $\alpha \subset \{1,2,\dots,n\}$, we denote by $|\alpha|$ the number of elements of α and by $A[\alpha]$ the principal submatrix of $A=(a_{ij})$ whose entries are a_{ij} for $i,j\in\alpha$. Let $\lambda_i(A)$, $i=1,2,\dots,n$, be the eigenvalues of A. The interior and closure of a subset X of M_n will be denoted by $\mathrm{Int}\, X$ and $\overline{X}$, respectively.

In order to discuss our main theorem, two lemmas are needed. The proof of the first lemma can be found in [2].

Lemma 1. *Let* $A\in\mathrm{SSM}_n$. *Then* $A[\alpha]\in\mathrm{SSM}^0_{|\alpha|}$ *for any* $\alpha\subset\{1,2,\dots,n\}$.

Since the eigenvalues of a matrix depend continuously on its entries [6], the following lemma can easily be proved.

Lemma 2. *With respect to the usual topology,*

(1) SM_n *and* SM_n^+ *are open, and*
(2) SM_n^0 *is closed.*

We also note that the assumption $A \in \mathrm{SSM}_n$ in Lemma 1 can be weakened to $A \in \mathrm{SSM}_n^0$.

With the above notation, our main theorems about the closure and interior of the set of strongly stable matrices in M_n are as follows.

THEOREM 1. $\overline{\mathrm{SSM}_n} = \mathrm{SSM}_n^0$.

Proof. Let $A \in \mathrm{SSM}_n^0$. Then the matrices

$$A^{(m)} = A + \frac{1}{m} I, \qquad m = 1, 2, \dots,$$

belong to SSM_n. Therefore, $A = \lim A^{(m)} \in \overline{\mathrm{SSM}_n}$.

Conversely, let $B \in \overline{\mathrm{SSM}_n}$. Then there exists a sequence $\{B^{(m)}\} \subset \mathrm{SSM}_n$ with $\lim B^{(m)} = B$. Therefore, for any nonnegative diagonal matrix D, $B^{(m)} + D \in \mathrm{SM}_n \subset \mathrm{SM}_n^0$. By Lemma 2, $B + D \in \mathrm{SM}_n^0$. Thus $B \in \mathrm{SSM}_n^0$, i.e. $\overline{\mathrm{SSM}_n} \subset \mathrm{SSM}_n^0$. ∎

THEOREM 2. $\mathrm{Int}\,\mathrm{SSM}_n = \mathrm{SSM}_n^+$.

Proof. First, we prove that $\mathrm{Int}\,\mathrm{SSM}_n \subset \mathrm{SSM}_n^+$. Let $A = (a_{ij}) \in \mathrm{Int}\,\mathrm{SSM}_n$. A sequence in M_n can be defined by

$$A^{(m)} = A - \frac{1}{m} I, \qquad m = 1, 2, \dots,$$

where I is the identity matrix. Obviously, $A^{(m)} \to A$ as $m \to \infty$. Since $A \in \mathrm{Int}\,\mathrm{SSM}_n$, there exists $N_1 > 0$ such that $A^{(m)} \in \mathrm{Int}\,\mathrm{SSM}_n \subset \mathrm{SSM}_n$ for $m > N_1$. By Lemma 1, the principal submatrices of $A^{(m)}$ are strongly semistable, i.e., $A^{(m)}[\alpha] + D_{|\alpha|}$ is semistable for any $\alpha \subset \{1, 2, \dots, n\}$ and for any nonnegative diagonal matrix $D_{|\alpha|} \in \mathrm{M}_{|\alpha|}$. Hence, $A[\alpha] + D_{|\alpha|} = A^{(m)}[\alpha] + D_{|\alpha|} + (1/m) I_{|\alpha|}$ is stable, i.e., $A \in \mathrm{SSM}_n^+$.

Conversely, we prove the reverse containment. Suppose to the contrary that $\mathrm{SSM}_n^+ \not\subset \mathrm{Int}\,\mathrm{SSM}_n$. Let $B \in \mathrm{SSM}_n^+ \subset \mathrm{SSM}_n$ and $B \notin \mathrm{Int}\,\mathrm{SSM}_n$. Then there exists a sequence $\{B^{(m)}\} \to B$ as $m \to \infty$ and $B^{(m)} \notin \mathrm{SSM}_n$ for all m, i.e., there exists a sequence $D^{(m)}$ of nonnegative diagonal matrices such that $B^{(m)} + D^{(m)} \notin \mathrm{SM}_n$ for all m, where $D^{(m)} = \mathrm{diag}(d_1^{(m)}, d_2^{(m)}, \dots, d_n^{(m)})$. Since $B \in \mathrm{SSM}_n \subset \mathrm{SM}_n$, there exists, by Lemma 2, $N_2 > 0$ such that $B^{(m)} \in \mathrm{SM}_n$ for all $m > N_2$. Without loss of generality, we can assume that the sequence

$\{D^{(m)}\}$ is convergent (finite or infinite). Otherwise, we consider its convergent subsequence. Now the problem can be discussed for two different cases as follows.

Case 1.

$$\lim_{m\to\infty} D^{(m)} = D.$$

In this case we obtain

$$\lim_{m\to\infty} \left(B^{(m)} + D^{(m)}\right) = B + D,$$

where $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$ is a nonnegative diagonal matrix. Since $B \in \mathrm{SSM}_n$, we have $B + D \in \mathrm{SM}_n$. By Lemma 2, there exists $N_3 > N_2 > 0$ such that $B^{(m)} + D^{(m)} \in \mathrm{SM}_n$ for $m > N_3$.

Case 2.

$$\lim_{m\to\infty} d_i^{(m)} = \begin{cases} \infty, & i \in \beta \subset \{1,2,\dots,n\}, \\ d_i, & i \notin \beta, \end{cases}$$

where β is nonempty. For the sake of convenience, we assume that $\beta = \{1,2,\dots,|\beta|\}$. Then $B^{(m)}, B$, and $D^{(m)}$ can be partitioned as follows:

$$B^{(m)} = \begin{pmatrix} B_{11}^{(m)} & B_{12}^{(m)} \\ B_{21}^{(m)} & B_{22}^{(m)} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \qquad D^{(m)} = \begin{pmatrix} D_1^{(m)} & 0 \\ 0 & D_2^{(m)} \end{pmatrix}.$$

where $B_{11}^{(m)}, B_{11}, D_1^{(m)} \in \mathrm{M}_{|\beta|}$. Let

$$C^{(m)} = B^{(m)} + D^{(m)} = \begin{pmatrix} B_{11}^{(m)} + D_1^{(m)} & B_{12}^{(m)} \\ B_{21}^{(m)} & B_{22}^{(m)} + D_2^{(m)} \end{pmatrix} = \begin{pmatrix} C_{11}^{(m)} & C_{12}^{(m)} \\ C_{21}^{(m)} & C_{22}^{(m)} \end{pmatrix},$$

where $D_1^{(m)} = \operatorname{diag}(d_1^{(m)}, d_2^{(m)}, \dots, d_{|\beta|}^{(m)})$ and $D_2^{(m)} = \operatorname{diag}(d_{|\beta|+1}^{(m)}, d_{|\beta|+2}^{(m)}, \dots, d_n^{(m)})$. Since we have for $i \in \beta$ that $d_i^{(m)} \to \infty$ as $m \to \infty$, it follows from Gerschgorin's theorem that

$$\operatorname{Re} \lambda_i(C^{(m)}) \to \infty, \quad i = 1,2,\dots,|\beta|, \qquad \text{as} \quad m \to \infty,$$

and furthermore

$$C_{11}^{(m)-1} \to 0 \qquad \text{as} \quad m \to \infty.$$

$B \in \mathrm{SSM}_n^+$ implies that $B_{22} + D_2 \in \mathrm{SM}_{n-|\beta|}$ for any nonnegative diagonal matrix $D_2 \in \mathrm{M}_{n-|\beta|}$. Therefore, $B_{22} + D_2$ is nonsingular, where $D_2 = \mathrm{diag}(d_{|\beta|+1}, d_{|\beta|+2}, \dots, d_n)$. Since $C_{22}^{(m)} - C_{21}^{(m)} C_{11}^{(m)-1} C_{12}^{(m)} \to B_{22} + D_2$ as $m \to \infty$, there exists by Lemma 2 $N_4 > N_3 > 0$ such that $C_{22}^{(m)} - C_{21}^{(m)} C_{11}^{(m)-1} C_{12}^{(m)} \in \mathrm{SM}_{n-|\beta|}$ for all $m > N_4$. Hence, $C_{22}^{(m)} - C_{21}^{(m)} C_{11}^{(m)-1} C_{12}^{(m)}$ is nonsingular for $m > N_4$. It follows that $C^{(m)}$ is also nonsingular for $m > N_4$, with inverse given by

$$C^{(m)-1} = \begin{pmatrix} C_{11}^{(m)-1} + C_{11}^{(m)-1} C_{12}^{(m)} Y^{(m)-1} C_{21}^{(m)} C_{11}^{(m)-1} & -C_{11}^{(m)-1} C_{12}^{(m)} Y^{(m)-1} \\ -Y^{(m)-1} C_{21}^{(m)} C_{11}^{(m)-1} & Y^{(m)-1} \end{pmatrix},$$

where

$$Y^{(m)} = C_{22}^{(m)} - C_{21}^{(m)} C_{11}^{(m)-1} C_{12}^{(m)}.$$

Obviously, we have

$$C^{(m)-1} \to \begin{pmatrix} 0 & 0 \\ 0 & (B_{22} + D_2)^{-1} \end{pmatrix} \qquad \text{as} \quad m \to \infty.$$

Since the eigenvalues $\lambda_i((C^{(m)})^{-1})$, $i = |\beta| + 1, \dots, n$, tend to the eigenvalues of $(B_{22} + D_2)^{-1}$, the eigenvalues $\lambda_i(C^{(m)})$, $i = |\beta| + 1, \dots, n$, tend to the eigenvalues of $B_{22} + D_2$, which have positive real parts, and therefore there exist $\delta > 0$, $M > 0$, and $N_5 > N_4 > 0$ such that

$$M > \mathrm{Re}\,\lambda_i(C^{(m)}) > \delta, \qquad i = |\beta| + 1, \dots, n,$$

for all $m > N_5$.

Thus, $C^{(m)} = B^{(m)} + D^{(m)}$ has $|\beta|$ eigenvalues whose real parts approach positive infinity as $m \to \infty$ and $n - |\beta|$ eigenvalues whose real parts are strictly positive and bounded for $m > N_5$, i.e., $B^{(m)} + D^{(m)} \in \mathrm{SM}_n$ for all $m > N_5$.

In both cases, $B^{(m)} + D^{(m)} \in \mathrm{SM}_n$ for $m > N_5$, which is contrary to the hypothesis. Therefore, $\mathrm{SSM}_n^+ \subset \mathrm{Int}\,\mathrm{SSM}_n$. ■

For D-stable matrices, it was mentioned in [1] that A is in the interior of the set of D-stable matrices if and only if the principal submatrices of A and $(A[\alpha])^{-1}$ are D-stable for every nonempty $\alpha \subset \{1,2,\dots,n\}$. Some equivalent conditions were presented there.

Hartfiel [4] gave an example as follows:

$$A = \begin{pmatrix} 1 & 0 & -50 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

which shows that the set of Volterra-Lyapunov stable matrices (a matrix A is said to be Volterra-Lyapunov stable if there exists a positive diagonal matrix D such that $AD + DA^T$ is positive definite) is not the interior of the set of D-stable matrices. We also note that since for $n \leqslant 3$ a D-stable matrix is strongly stable [2], this example shows that the set of Volterra-Lyapunov stable matrices is not the interior of the set of strongly stable matrices.

The discussion in this paper is restricted in real matrices. The extension to complex matrices is easy.

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